Complex Impedance

Complex Impedance

- The **Impedance Z** of a circuit is the ratio of phasor voltage V to the phasor current I **(Ohm's Law)** .which is usually represented by complex number.

$$Z = \frac{\mathbf{V}}{\mathbf{I}} \quad \text{or} \quad \mathbf{V} = Z\mathbf{I}$$ (Ohm's Law)

- This **Impedance Z** is usually a complex number and not a sine wave.

Complex Impedance in case of Resistace Z_R

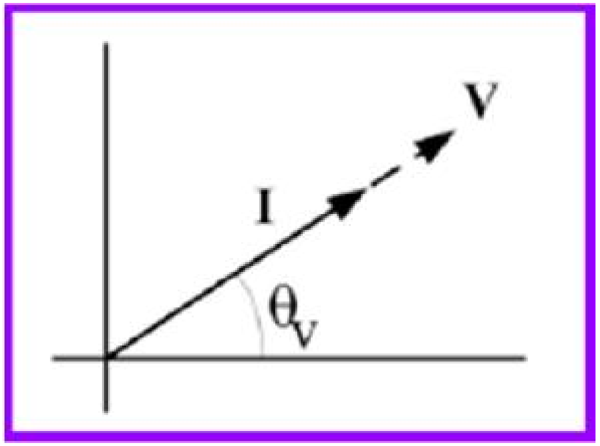
- **In the Resistance circuit, the phasor current I is in phase with phasor voltage V.**

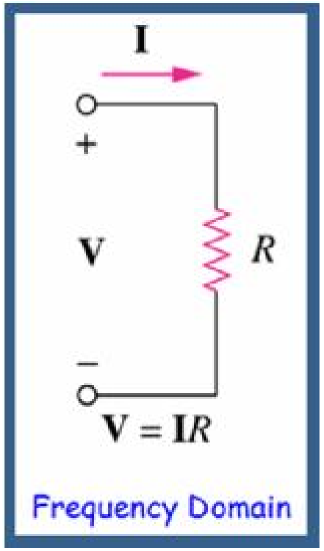
$$\mathbf{V} = V\angle\theta_V \qquad \mathbf{I} = I\angle\theta_V$$ (current in phase with voltage)

Appling Ohm's law:

$$Z_R = \frac{V\angle\theta_V}{I\angle\theta_V} = \frac{V}{I}\angle 0^\circ$$

$$Z_R = R\angle 0^\circ = R$$

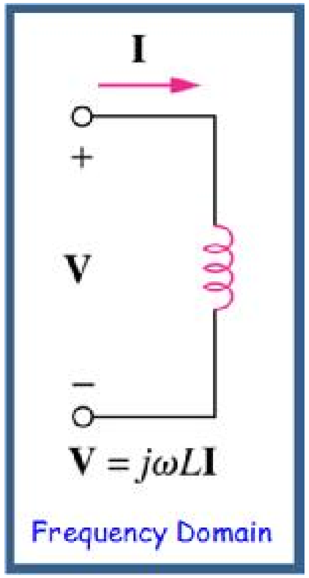




Complex Impedance in case of Inductive reactance Z_L

- **In the Inductance circuit, the phasor current I is LAGs voltage V by 90º.**

$V = V\angle\theta_V \quad I = I\angle(\theta_V - 90)$ (current lags voltage by 90º)

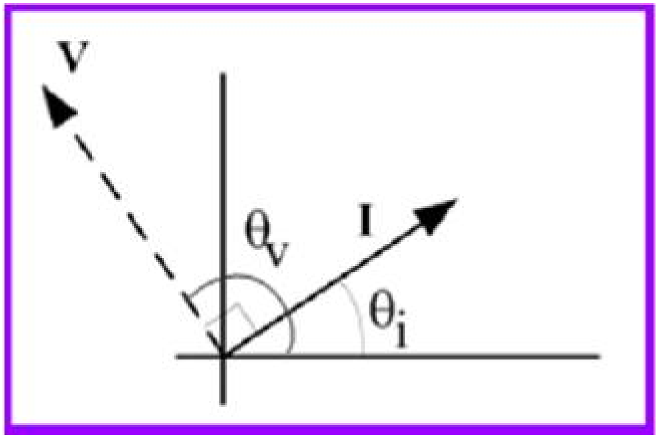


Appling Ohm's law:

$$Z_L = \frac{V\angle\theta_V}{I\angle(\theta_V - 90)}$$

$$= \frac{V}{I}\angle 90^\circ$$

$$Z_L = X_L\angle 90^\circ = jX_L$$



Complex Impedance in case of Capacitive reactance Z_c

- **In the Capacitive circuit, the phasor current I is LEADs voltage V by 90º.**

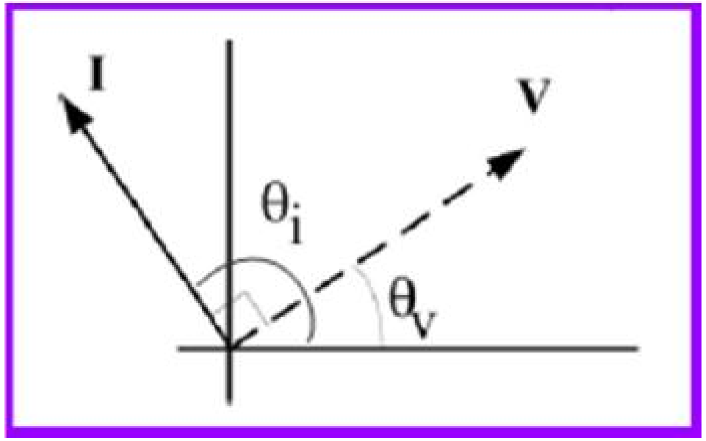
$$V = V\angle\theta_V \quad I = I\angle(\theta_V + 90)$$ (current leads voltage by 90º)

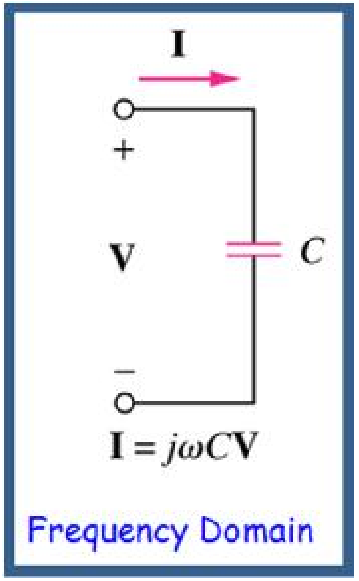
Appling Ohm's law:

$$Z_C = \frac{V\angle\theta_V}{I\angle(\theta_V + 90)}$$

$$= \frac{V}{I}\angle -90^\circ$$

$$Z_C = X_C\angle -90^\circ = -jX_C$$





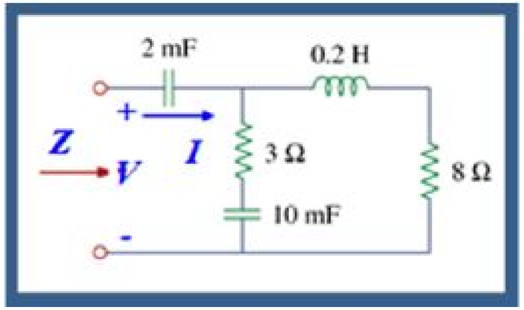
Impedance of Joint Elements

- The **Impedance Z** **represents the opposition of the circuit to the flow of sinusoidal current.**

$$Z = \frac{V}{I} = R + jX =$$

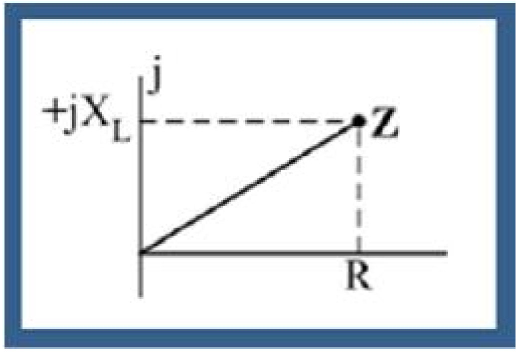
$$= \text{Resistance} + j \times \text{Reactance}$$

$$= |Z| \angle \theta$$



$$|Z| = \sqrt{R^2 + X^2} \qquad \theta = \tan^{-1}\frac{X}{R}$$

$$R = |Z|\cos\theta \qquad X = |Z|\sin\theta$$

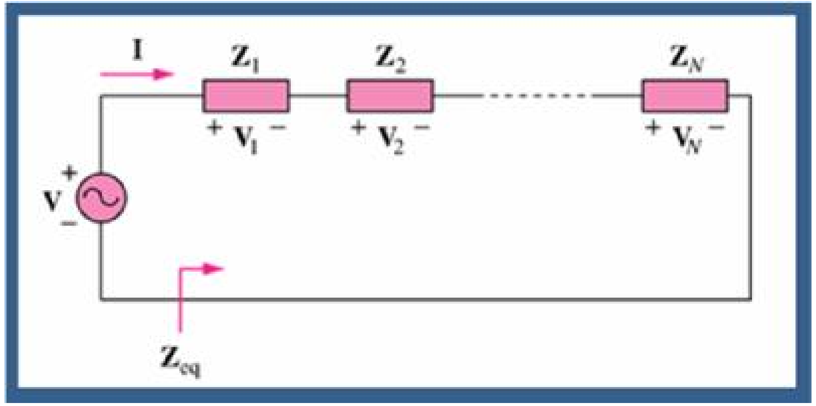


- **The Reactance is Inductive if *X* is positive and it is Capacitive if *X* is negative.**

Series and Parallel Circuits

Series Circuits

- **The Kirchoff"s Voltage Law (KVL) holds in the frequency domain. For series connected impedances:**



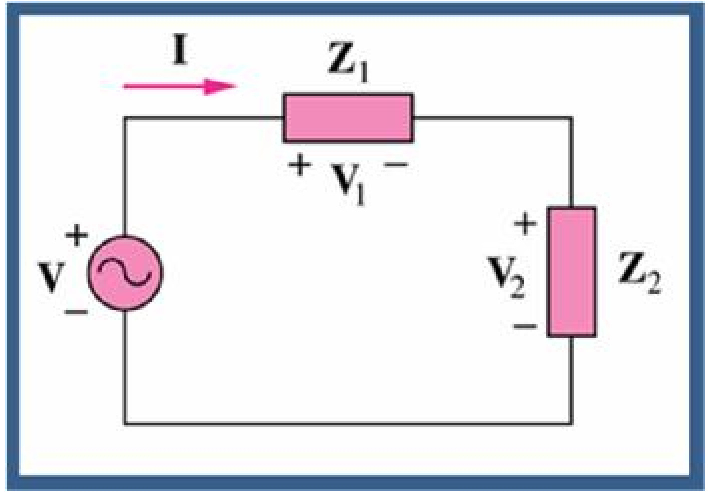
Current is Constant

$$\mathbf{V} = \mathbf{V}_1 + \mathbf{V}_2 + \cdots + \mathbf{V}_N = \mathbf{I}(\mathbf{Z}_1 + \mathbf{Z}_2 + \cdots + \mathbf{Z}_N)$$

$$\mathbf{Z}_{eq} = \frac{\mathbf{V}}{\mathbf{I}} = \mathbf{Z}_1 + \mathbf{Z}_2 + \cdots + \mathbf{Z}_N \quad \text{(Equivalent Impedance)}$$

Voltage Divider rule

- The **Voltage Division** for two elements in series is:



$$I = \frac{V}{Z_1 + Z_2}$$

Since $V_1 = Z_1 I$ and $V_2 = Z_2 I$, then

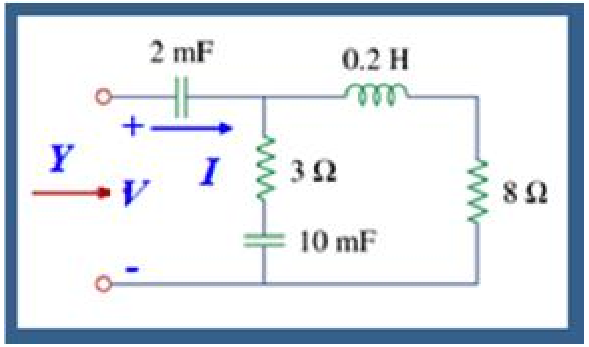
$$V_1 = \frac{Z_1}{Z_1 + Z_2} V$$

$$V_2 = \frac{Z_2}{Z_1 + Z_2} V$$

Admittance of Joint Elements

- The **Admittance Y** represents the admittance of the circuit to the flow of sinusoidal current. The admittance is measured in **Siemens (s)**

$$Y = \frac{1}{Z} = \frac{I}{V} = G + jB$$



$$= \text{Conductance} + j\,\text{Suseptance} = |Y| \angle \theta$$

$$Y = G + jB = \frac{1}{R + jX} \frac{R - jX}{R - jX} = \frac{R - jX}{R^2 + X^2}$$

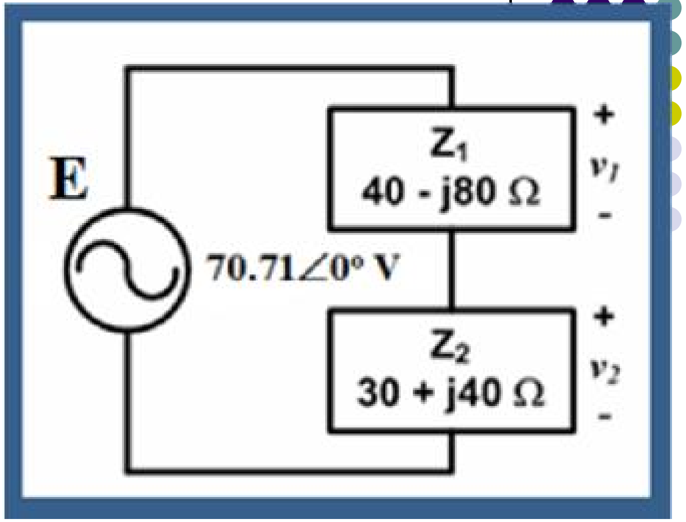
$$G = \frac{R}{R^2 + X^2}$$

$$B = -\frac{X}{R^2 + X^2}$$

Ex.5: consider the circuit,

(a) Calculate the sinusoidal voltages v_1 and v_2 using phasors and voltage divider rule

(b) Sketch the phasor diagram showing E, V_1 and V_2.



Solution

(a) the sinusoidal voltages *v*1 and *v*2 using phasors

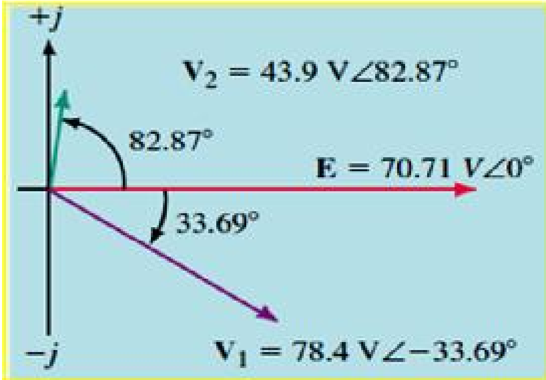
$$V_1 = \frac{Z_1}{Z_1 + Z_2} V$$

$$V_2 = \frac{Z_2}{Z_1 + Z_2} V$$

$$V_1 = \left(\frac{40 - j80}{(40 - j80) + (30 + j40)}\right)(70.71\angle 0^o) = 78.4\angle -33.69^o V$$

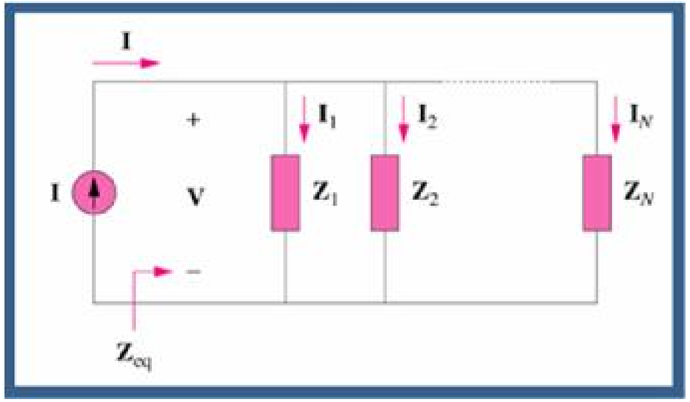
$$V_2 = \left(\frac{30 + j40}{(40 - j80) + (30 + j40)}\right)(70.71\angle 0^o) = 43.9\angle 82.87^o V$$

(b) Phasor diagram



Parallel Circuits

- **The Kirchoff"s Current Law (KCL) holds in the frequency domain. For Parallel connected impedances:**



Voltage is Constant

$$I = I_1 + I_2 + \cdots + I_N = V\left(\frac{1}{Z_1} + \frac{1}{Z_2} + \cdots + \frac{1}{Z_N}\right)$$

(Eqiv. Admitance)

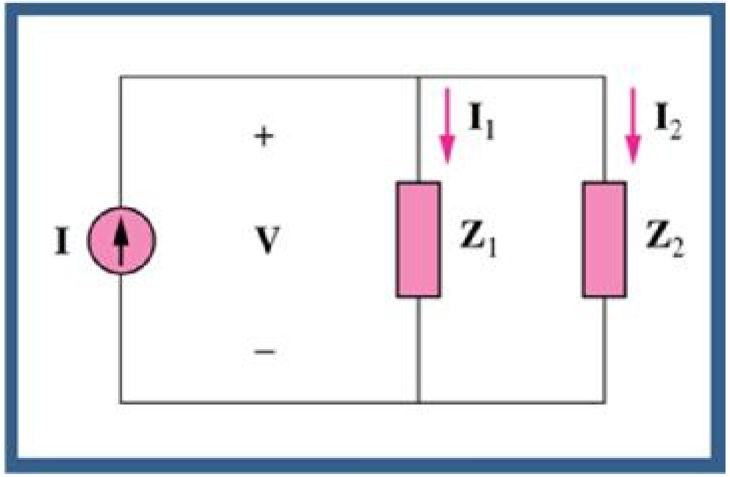
$$\frac{1}{Z_{eq}} = \frac{I}{V} = \frac{1}{Z_1} + \frac{1}{Z_2} + \cdots + \frac{1}{Z_N} \qquad Y_{eq} = Y_1 + Y_2 + \cdots + Y_N$$

In case of two resistors

$$Z_{eq} = \frac{1}{Y_{eq}} = \frac{1}{Y_1 + Y_2} = \frac{1}{1/Z_1 + 1/Z_2} = \frac{Z_1 Z_2}{Z_1 + Z_2}$$

Current Divider rule

- The **Current Division** for two elements is:

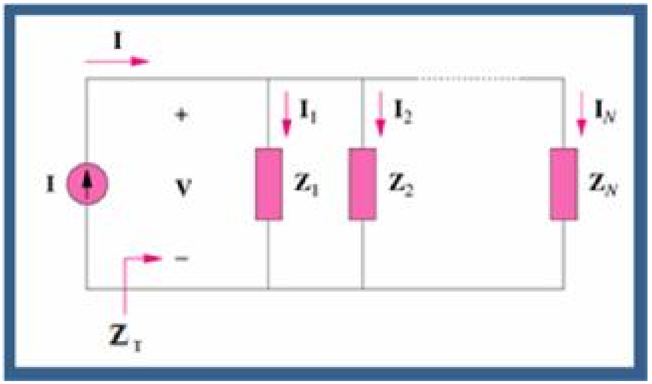


$$Z_{eq} = \frac{1}{Y_{eq}} = \frac{1}{Y_1 + Y_2} = \frac{1}{1/Z_1 + 1/Z_2} = \frac{Z_1 Z_2}{Z_1 + Z_2}$$

$$V = IZ_{eq} = I_1 Z_1 = I_2 Z_2$$

$$I_1 = \frac{Z_2}{Z_1 + Z_2} I \qquad I_2 = \frac{Z_1}{Z_1 + Z_2} I$$

- The **Current Division** for more elements is:



$$I_{Z_N} = \frac{Z_T}{Z_N} I_T$$

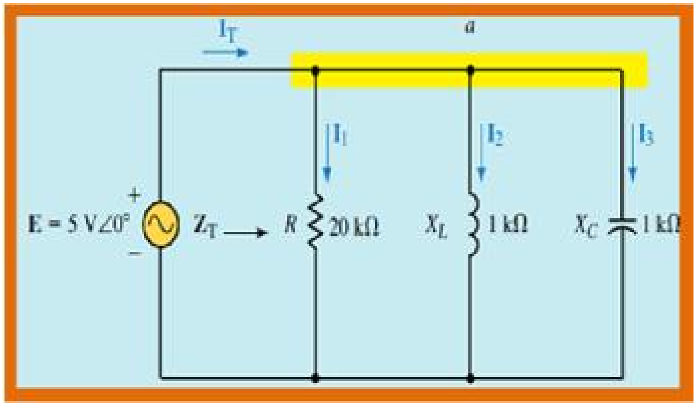
Ex. 6: Refer to the circuit;

(a) Find the total impedance, Z_T.

(b) Determine the supply current, I_T.

(c) Calculate I_1, I_2, using current divider rule.

(d) Verify Kirchhoff's current law at node a.



Solution

$$a.\quad \frac{1}{Z_T} = \frac{1}{20} + \frac{1}{j1} + \frac{1}{-j1} = \frac{1}{20} \qquad Z_T = 20\angle 0^o\, k\Omega$$

$$Z_R = 20\angle 0^o\, k\Omega$$

$$Z_L = 1\angle 90^o\, k\Omega$$

$$Z_C = 1\angle -90^o\, k\Omega$$

$$b.\quad I_T = \frac{V}{Z_T} = \frac{5\angle 0}{20x10^3\angle 0} = 250x10^{-6}\angle 0^o\, A = 250\angle 0^o\, mA$$

$$c.\quad I_1 = \frac{20\angle 0}{20\angle 0} x250\angle 0^o = 250\angle 0^o\, \mu A$$

$$I_2 = \frac{20\angle 0}{1\angle 90} x250\angle 0^o = 5000\angle -90^o\, \mu A$$

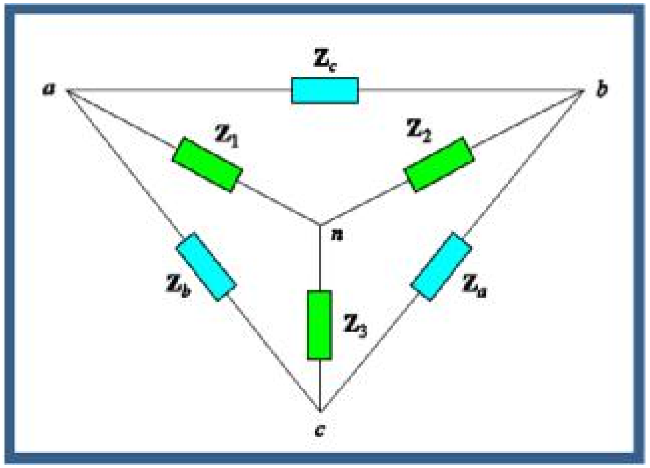
$$I_3 = \frac{20\angle 0}{1\angle -90} x250\angle 0^o = 5000\angle 90^o\, \mu A$$

$$d.\quad I_T = I_1 + I_2 + I_3 = 250\angle 0 + 5000\angle \text{-}90 + 5000\angle 90 = 250\angle 0^o\, \mu A$$

$\therefore$ Kirchhoff's current law is verified.

Delta to Star and Star to Delta Conversions

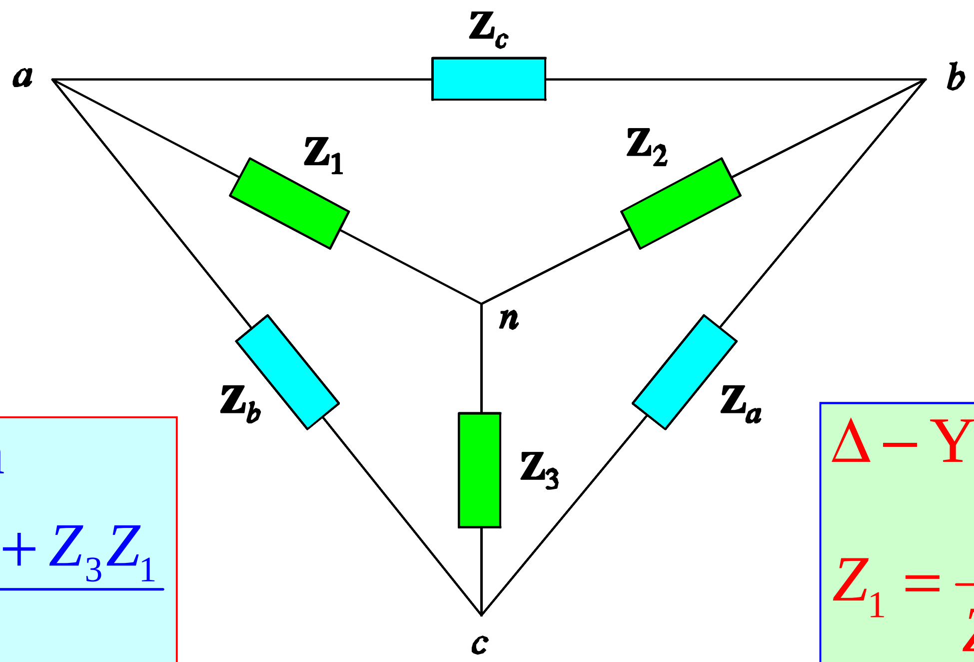
Y-Δ and Δ-Y Equivalent Circuits



- Y-Δ and Δ-Y type equivalent conversions will be most useful when considering Three Phase circuits.
- Impedances Z_1, Z_2 and Z_3 are Y connected.
- Impedances Z_a, Z_b and Z_c are Δ connected.
- Y and Δ forms can be equivalently converted from one form to the other.
- Y-Δ and Δ-Y conversions are valid for impedances as well as resistive circuits.
- Y-Δ and Δ-Y type equivalent conversions will be most useful when considering Three Phase circuits.

Y-Δ and Δ-Y Equivalent Circuits

- Y-Δ and Δ-Y type equivalent conversions will be useful when considering Three Phase circuits.



Y − Δ Conversion

$$Z_a = \frac{Z_1Z_2 + Z_2Z_3 + Z_3Z_1}{Z_1}$$

$$Z_b = \frac{Z_1Z_2 + Z_2Z_3 + Z_3Z_1}{Z_2}$$

$$Z_c = \frac{Z_1Z_2 + Z_2Z_3 + Z_3Z_1}{Z_3}$$

Δ − Y Conversion

$$Z_1 = \frac{Z_bZ_c}{Z_a + Z_b + Z_c}$$

$$Z_2 = \frac{Z_aZ_c}{Z_a + Z_b + Z_c}$$

$$Z_3 = \frac{Z_aZ_b}{Z_a + Z_b + Z_c}$$

Balanced Y-Δ and Δ-Y Equivalent Circuits

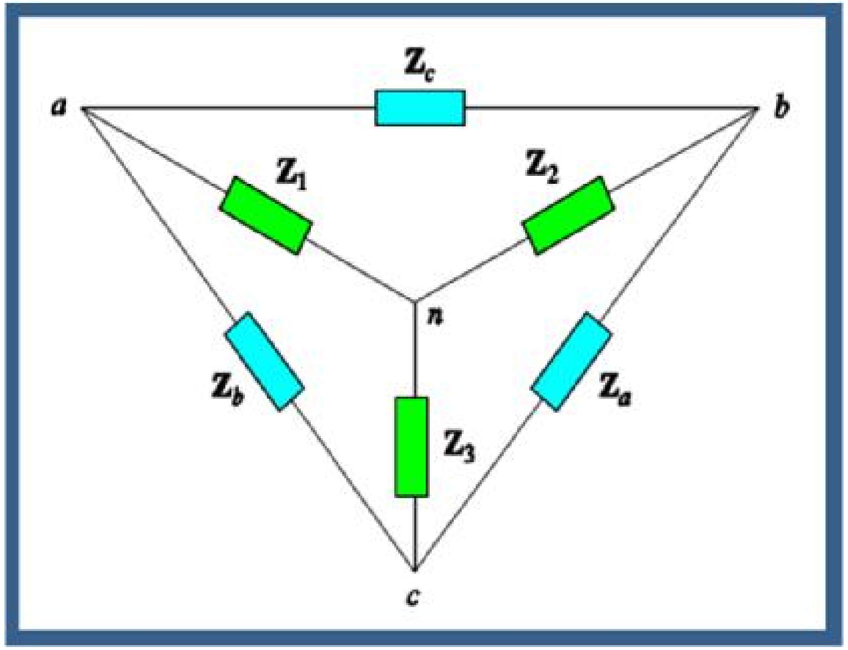
- **A delta (Δ) or Y (wye) circuit is balanced if it has equal impedances in all three branches.**
- **Y-Δ and Δ-Y conversions is very simple for balanced circuits.**

Balanced Impedance Conversions:

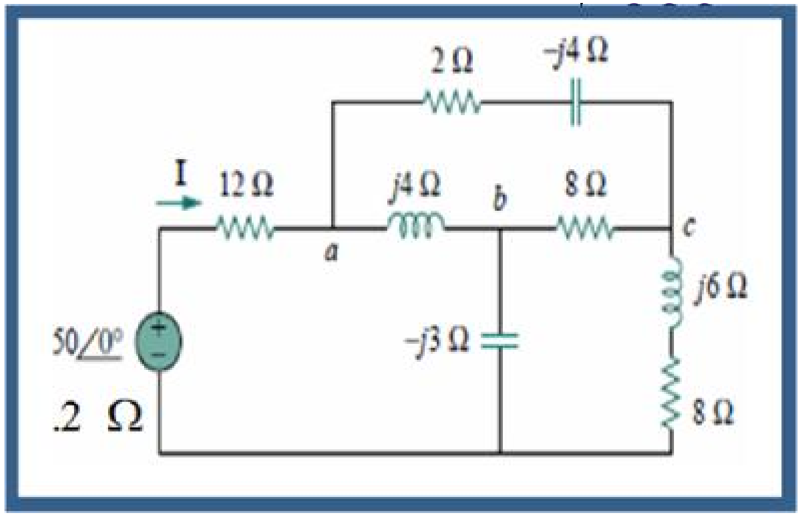
$$Z_Y = Z_1 = Z_2 = Z_3$$

$$Z_\Delta = Z_a = Z_b = Z_c$$

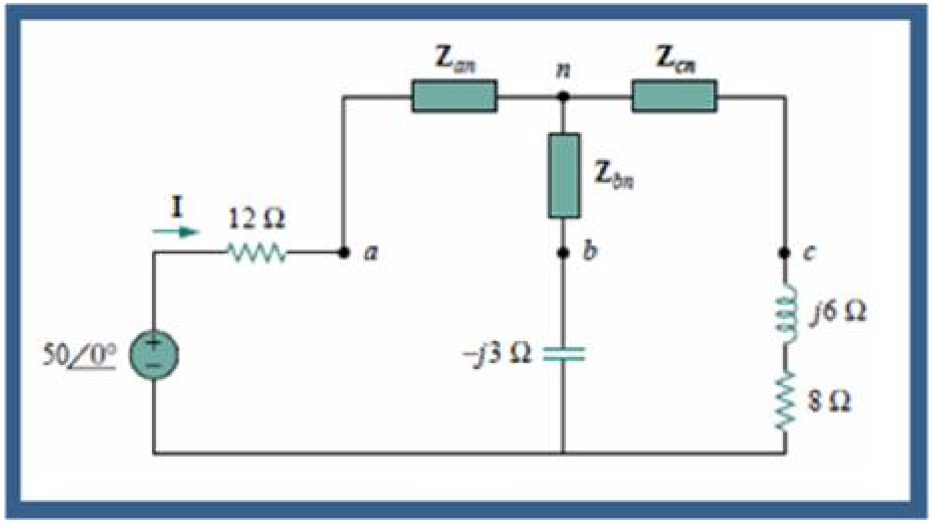
$$Z_\Delta = 3Z_Y \qquad Z_Y = \frac{1}{3} Z_\Delta$$



Ex. 7: Find the current I in the circuit.



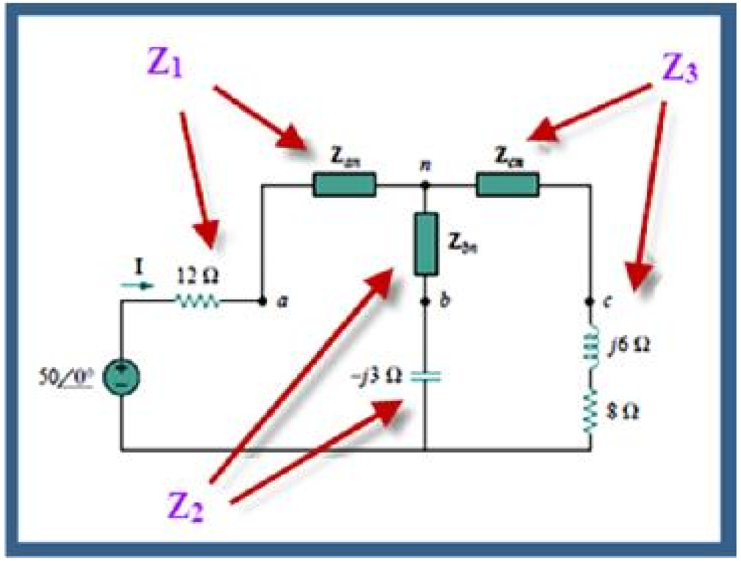
Solution

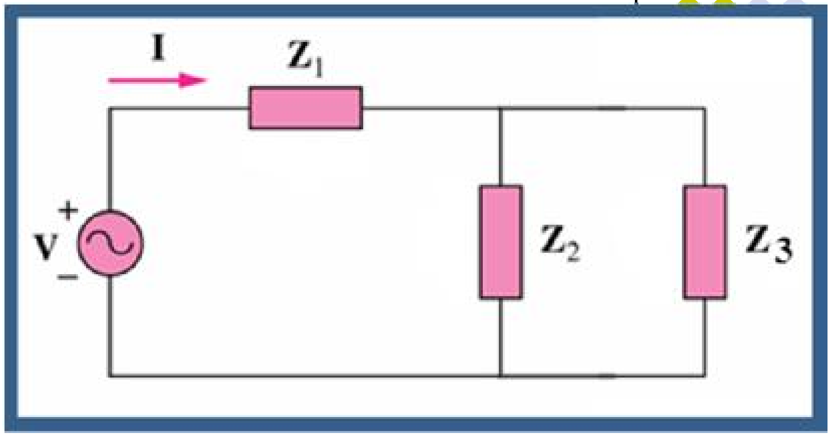


$$Z_{an} = \frac{j4(2-j4)}{j4+2-j4+8} = \frac{16+j8}{10} = 1.6 + j0.8\ \Omega$$

$$Z_{bn} = \frac{j4(8)}{10} = j3.2\ \Omega \qquad Z_{cn} = \frac{8(2-j4)}{10} = 1.6 - j3.2\ \Omega$$

Ex. 7: Find the current I in the circuit.





$$Z_1 = Z_{an} + 12 = 13.6 + j0.8 \qquad Z_2 = Z_{bn} - j3 = j0.2$$

$$Z_3 = Z_{cn} + j6 + 8 = 9.6 + j2.8$$

$$Z_T = Z_1 + \frac{Z_2 Z_3}{Z_2 + Z_3} = 13.6 + j0.8 + \frac{j0.2(9.6 + j2.8)}{9.6 + j3} = 13.6 + j1 \ \Omega$$

The source current $I = \frac{V}{Z_T} = \frac{50\angle 0}{13.6 + j1} = 3.666\angle -4.2^o \ \Omega$

AC Bridges

AC Bridges

- The AC bridge is **Balanced when no current flows** through the meter. AC bridges are **used in measuring inductance and capacitance** values.

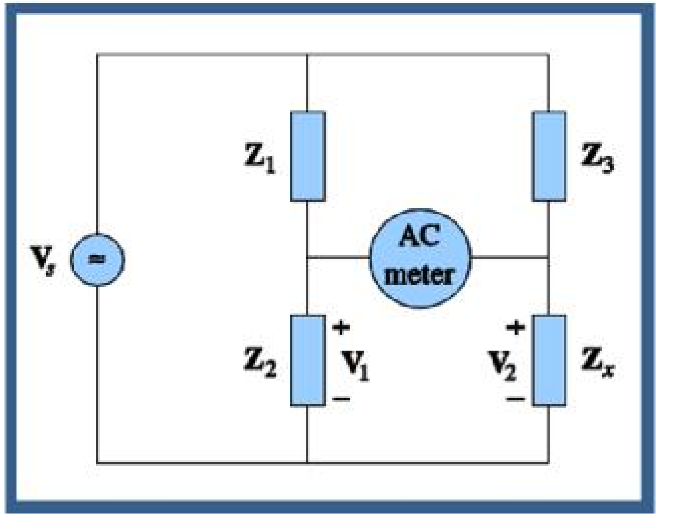
At balance case: $I_{meter} = 0$

$$V_1 = V_2$$

$$V_1 = \frac{Z_2}{Z_1 + Z_2} V_s = V_2 = \frac{Z_x}{Z_3 + Z_x} V_s$$

$$\frac{Z_2}{Z_1 + Z_2} = \frac{Z_x}{Z_3 + Z_x} \Rightarrow Z_2 Z_3 = Z_1 Z_x$$

$$\boxed{Z_x = \frac{Z_3}{Z_1} Z_2}$$

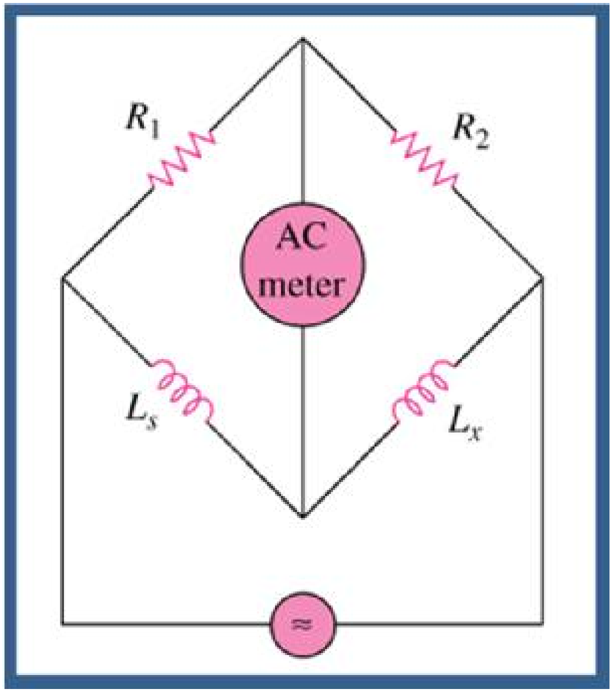


AC bridge circuit

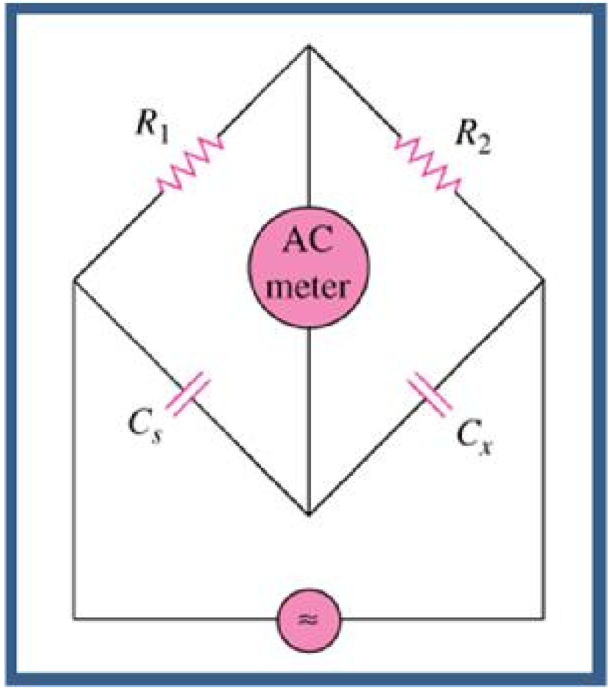
Z_x Unknown value necessary for balancing the bridge

AC Bridges

- Unknown capacitance and inductances C_x and L_x are measured in terms of the known standard values C_s and L_s



AC Bridge for measuring L



AC Bridge for measuring C.

$$L_x = \frac{R_2}{R_1} L_s$$

$$C_x = \frac{R_1}{R_2} C_s$$

Ex.8: **Determine the series components that make up Z_x for balancing the bridge. Assume Z_1 = 4.8 kΩ resistor, Z_2 is 10 Ω in series with 0.25 µH, Z_3 = 12 kΩ and f = 6 Mhz.**

Solution

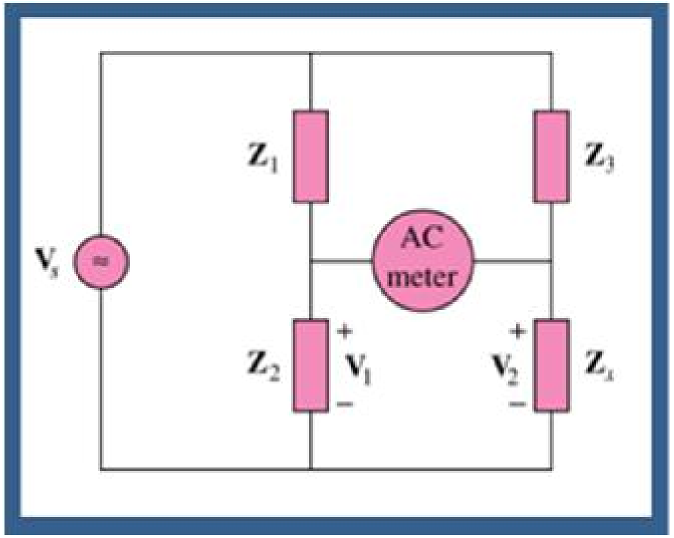
$$Z_x = (Z_3 / Z_1) Z_2 \qquad Z_1 = 4.8\ k\Omega$$

$$Z_3 = 12\ k\Omega$$

$$Z_2 = 10 + j\omega L$$

$$= 10 + j(2\pi)(6\times 10^6)(0.25\times 10^{-6})$$

$$= 10 + j9.425\ k\Omega$$



$$Z_x = \frac{12k}{4.8k}(10 + j9.425) = 25 + j23.5625\ \Omega$$

$$R_x = 25, \qquad X_x = 23.5625 = \omega L_x$$

$$L_x = \frac{X_x}{2\pi f} = \frac{23.5625}{2\pi(6\times 10^6)} = 0.625\ \mu H$$

The series components are 25 Ω with a 0.625 µH

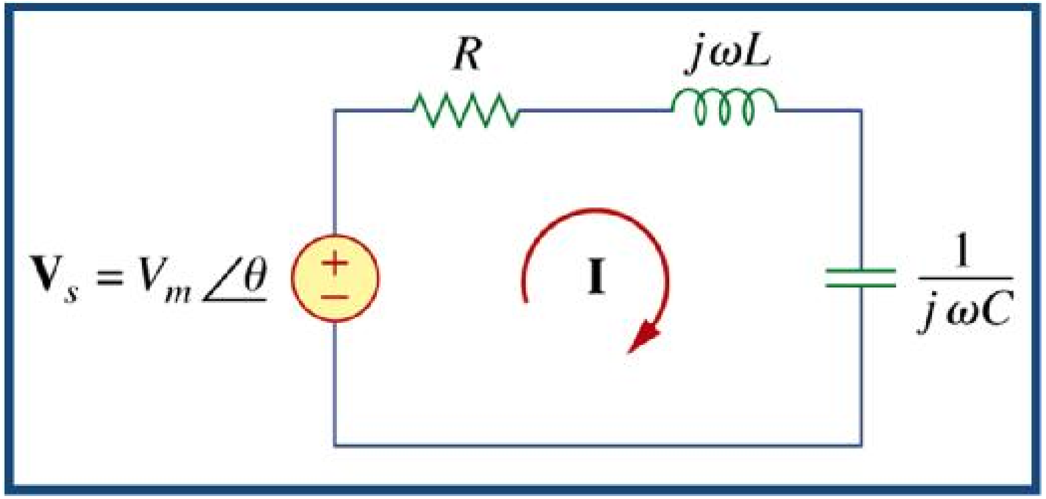
Resonance
in
AC Circuits

Resonance in AC Circuits

- Resonance is a condition in an RLC circuit in which the capacitive and reactive reactance are equal in magnitude, the result is a purely resistive impedance.

- Resonance circuits are useful for constructing filters and used in many application as signal processing and communications systems .

Series Resonant Circuit



- At resonance, the impedance consists only resistive component R.
- The value of current will be maximum since the total impedance is minimum.
- The voltage and current are in phase.
- Maximum power occurs at resonance since the power factor is unity.

Series Resonant Circuit

Total impedance of series RLC Circuit is

$$Z_{Total} = R + jX_L - jX_C$$

$$Z_{Total} = R + j(X_L - X_C)$$

At Resonance:

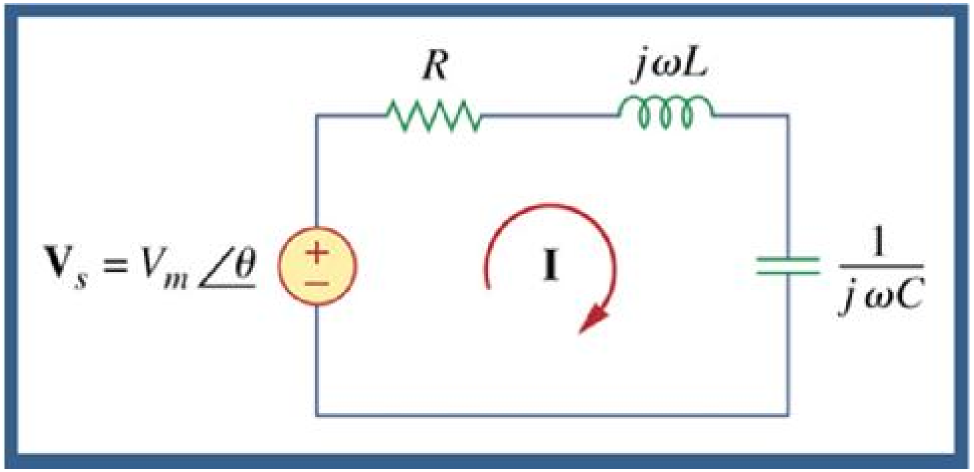
$$X_L = X_C$$

The impedance now reduce to

$$Z_{Total} = R$$

The current at resonance

$$I_m = \frac{V_s}{Z_{Total}} = \frac{V_m}{R}$$



The highest power is at ω_o

$$P(w_o) = I^2 R = \frac{V^2}{R}$$

Series Resonance

Resonance Frequency

Resonance frequency is the frequency where the condition of resonance occur.

$$\mathrm{Im}(z) = 0 \Rightarrow X_L = X_C$$

$$wL = \frac{1}{wC} \Rightarrow wL - \frac{1}{wC} = 0 \Rightarrow w_o L = \frac{1}{w_o C}$$

Resonance Frequency

$$\omega_o = \frac{1}{\sqrt{LC}}$$ rad/sec

$$f_o = \frac{1}{2\pi\sqrt{LC}}$$ Hz

Series Resonance

Half-power Frequency ω_1 & ω_2

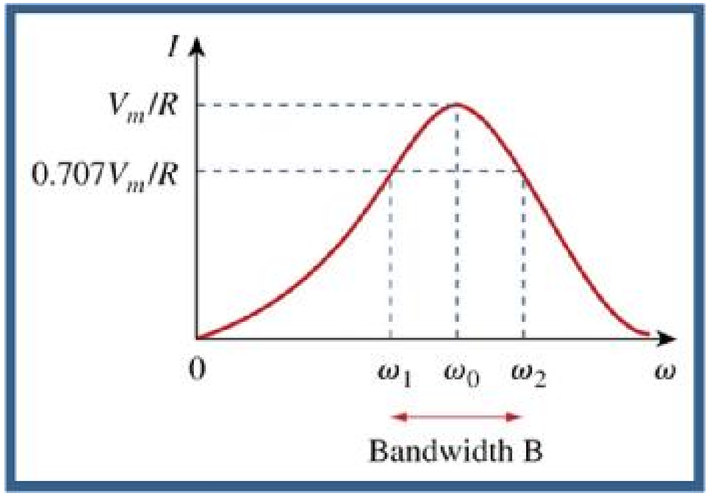
Half-power frequencies is the frequency when the magnitude of the output voltage or current is decrease by the factor of $1 / \sqrt{2}$ from its maximum value. Also known as cutoff frequencies.

$$I = \frac{I}{\sqrt{2}} = 0.707I$$

- The half-power frequencies ω_1 and ω_2 can be obtained by setting,

$$|Z(\omega_1)| = |Z(\omega_2)| = \sqrt{R^2 + (\omega L - \frac{1}{\omega C})^2} = \sqrt{2}R$$

$$P(\omega_1) = P(\omega_2) = \frac{\left(\frac{V_m}{\sqrt{2}}\right)^2}{2R}$$



Response curve

$$\omega_1 = -\frac{R}{2L} + \sqrt{\left(\frac{R}{2L}\right)^2 + \frac{1}{LC}}$$

$$\omega_2 = +\frac{R}{2L} + \sqrt{\left(\frac{R}{2L}\right)^2 + \frac{1}{LC}}$$

Series Resonance

Maximum Power Dissipated

The maximum current at resonance where:

$$I = \frac{V_m}{R}$$

Thus maximum power dissipated is:

$$P = I^2 R \quad \Rightarrow \quad \text{at } \omega = \omega_0$$

The average power dissipated by the RLC circuit is:

$$P = \frac{1}{2} I^2 R \qquad P = \frac{1}{2} \frac{V^2_m}{R} \quad \Rightarrow \quad \text{at } \omega = \omega_1,\ \omega_2,$$

Series Resonance

Bandwidth of resonant circuit, *b*

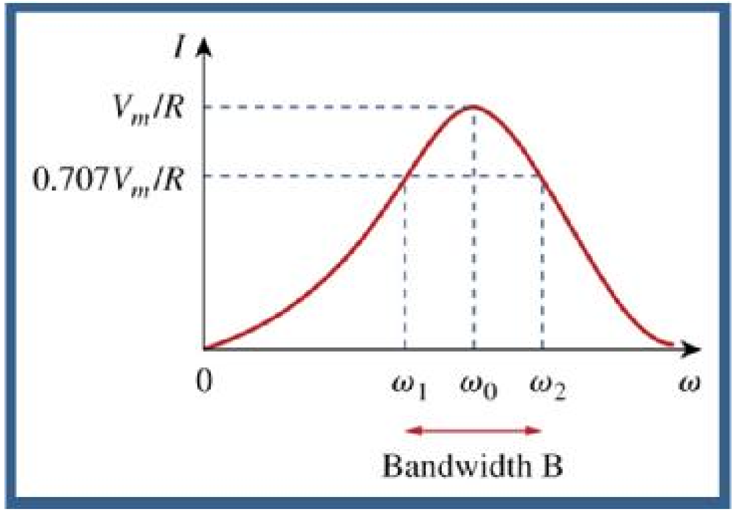
Bandwidth, *b* is define as the difference between the two half power frequencies. The width of the response curve is determine by the bandwidth.

$$B = \omega_2 - \omega_1$$

$$\beta = \frac{R}{L}$$

Resonance frequency can be obtained from the half-power frequencies.

$$\omega_o = \sqrt{\omega_1 \omega_2}$$



Response curve

Quality Factor (Q-Factor)

Quality factor is used to measure the "sharpness" of response curve

$$Q = 2\pi \frac{\text{Peak Energy Stored}}{\text{Energy Dissipated in one Period at Resonance}}$$

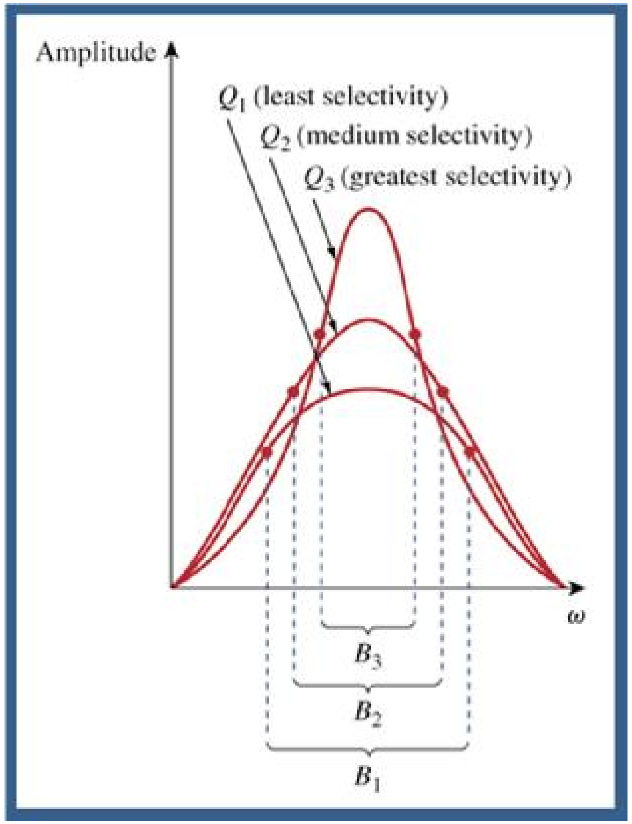
Quality factor is the ratio of resonance frequency to the bandwidth

$$Q = \frac{\omega_o}{\beta}$$

$$\beta = \frac{R}{L}$$

$$Q = \frac{\omega_o L}{R}$$

Selectivity defines how well a resonant circuit responds to certain frequencies.



- Higher value of Q, smaller the bandwidth. (Higher the selectivity)
- Lower value of Q, larger the bandwidth. (Lower the selectivity)

High - Q

It is to be a high-Q circuit when its quality factor is equal or greater than 10.

For a high-Q circuit ($Q \geqslant 10$), the half-power frequencies are, for all practical purposes, symmetrical around the resonant frequency and can be approximated as

$$\omega_1 \cong \omega_o - \frac{\beta}{2}$$

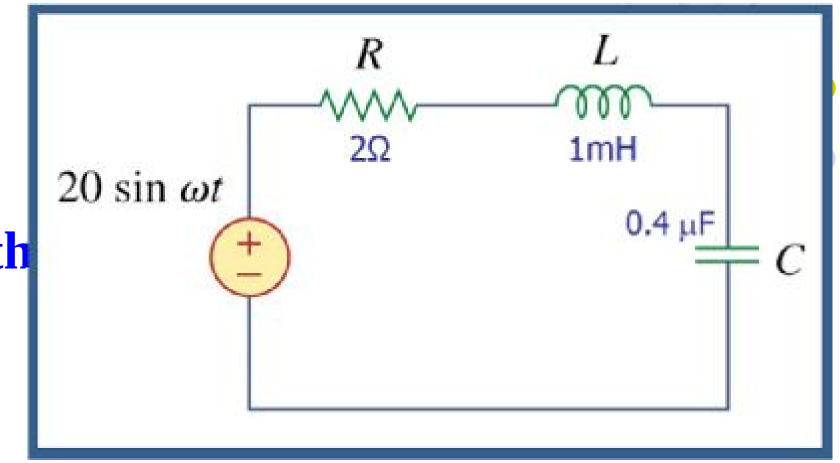
$$\omega_2 \cong \omega_o + \frac{\beta}{2}$$

Ex.9: For the circuit shown:

(a) Find the resonant frequency and the half power frequencies

(b) Calculate the quality factor and bandwidth

(c) Determine the amplitude of the current at ω_o, ω_1 and ω_2



Solution

(a)

$$\omega_0 = \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{10^{-3} \times 0.4 \times 10^{-6}}} = 50 \text{ krad/s}$$

$$\omega_1 = -\frac{R}{2L} + \sqrt{\left(\frac{R}{2L}\right)^2 + \frac{1}{LC}} = -\frac{2}{2 \times 10^{-3}} + \sqrt{(10^3)^2 + (50 \times 10^3)^2}$$

$$\omega_1 = -1 + \sqrt{1 + 2500} \text{ krad/s} = 49 \text{ krad/s} \qquad \omega_2 = 1 + \sqrt{1 + 2500} \text{ krad/s} = 51 \text{ krad/s}$$

(b)

$$Q = \frac{\omega_0}{B} = \frac{50}{2} = 25 \qquad B = \omega_2 - \omega_1 = 2 \text{ krad/s}$$

(c)

At $\omega = \omega_0$

$$I = \frac{V_m}{R} = \frac{20}{2} = 10 \text{ A}$$

At $\omega = \omega_1, \omega_2$

$$I = \frac{V_m}{\sqrt{2}R} = \frac{10}{\sqrt{2}} = 7.071 \text{ A}$$

Amplitude

$$i(t) = I_m \sin \omega t$$

Parallel Resonant Circuit

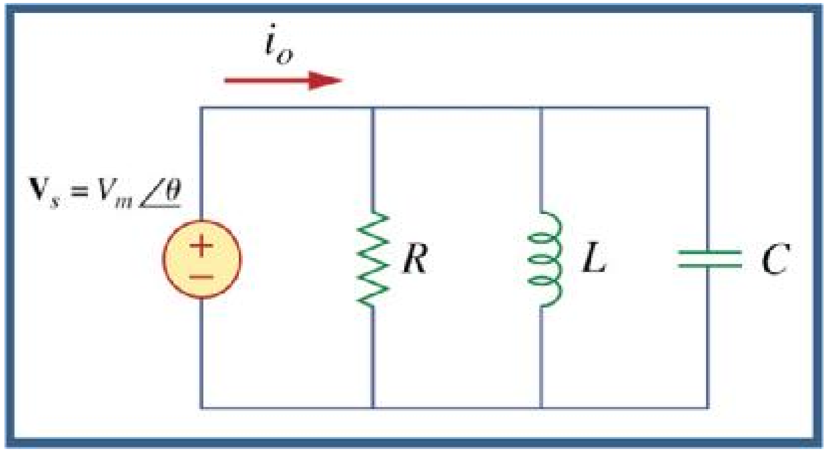
The total admittance:

$$Y_{Total} = Y_1 + Y_2 + Y_3$$

$$Y_{Total} = \frac{1}{R} + \frac{1}{(jwL)} + \frac{1}{(-j/wC)}$$

$$Y_{Total} = \frac{1}{R} + \frac{-j}{\omega L} + j\omega C$$

$$Y_{Total} = \frac{1}{R} + j(\omega C - 1/\omega L)$$



Resonance occur when:

$$\omega C = \frac{1}{\omega L}$$

- At resonance, the impedance consists only conductance G.
- The value of current will be minimum since the total admittance is minimum.
- The voltage and current are in phase.

Parameters in Parallel Circuit

Parallel resonant circuit has same parameters as the series resonant circuit.

Resonance frequency

$$\omega_o = \frac{1}{\sqrt{LC}} \text{ rad/s}$$

Half-power frequencies

$$\omega_1 = \frac{-1}{2RC} + \sqrt{\left(\frac{1}{2RC}\right)^2 + \left(\frac{1}{LC}\right)} \text{ rad/s}$$

$$\omega_2 = \frac{1}{2RC} + \sqrt{\left(\frac{1}{2RC}\right)^2 + \left(\frac{1}{LC}\right)} \text{ rad/s}$$

Bandwidth

$$\beta = \omega_2 - \omega_1 = \frac{1}{RC}$$

Q - Factor

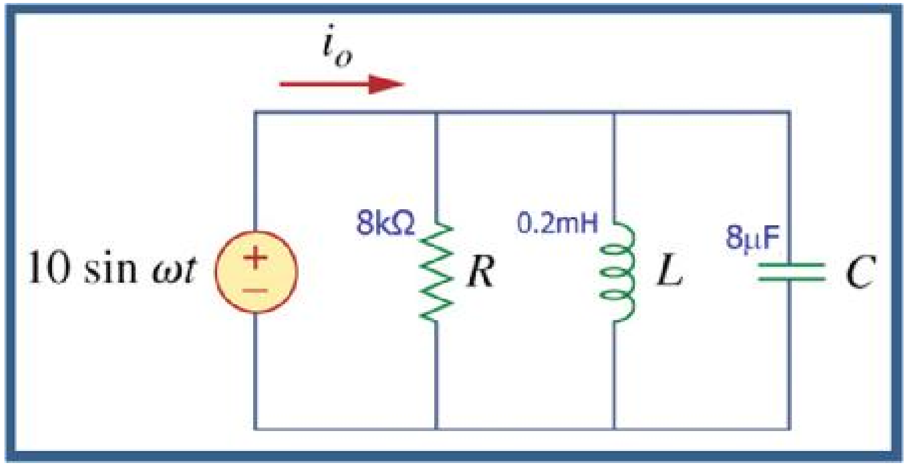
$$Q = \frac{\omega_o}{\beta} = \omega_o RC = \frac{R}{\omega_o L}$$

For a high-Q circuit ($Q \geqslant 10$)

$$\omega_1 \cong \omega_o - \frac{\beta}{2}$$

$$\omega_2 \cong \omega_o + \frac{\beta}{2}$$

Ex.10: For the circuit shown:
(a) Calculate ω_o, Q and B
(b) Find ω_1 and ω_2
(c) Determine the power dissipated at ω_o, ω_1 and ω_2



Solution

(a)

$$\omega_0 = \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{0.2\times 10^{-3}\times 8\times 10^{-6}}} = \frac{10^5}{4} = 25 \text{ krad/s}$$

$$Q = \frac{R}{\omega_0 L} = \frac{8\times 10^3}{25\times 10^3\times 0.2\times 10^{-3}} = 1600$$

$$B = \frac{\omega_0}{Q} = 15.625 \text{ rad/s}$$

(b)

$$\omega_1 = \omega_0 - \frac{B}{2} = 25{,}000 - 7.812 = 24{,}992 \text{ rad/s}$$

$$\omega_2 = \omega_0 + \frac{B}{2} = 25{,}000 + 7.8125 = 25{,}008 \text{ rad/s}$$

At $\omega = \omega_0$, $\mathbf{Y} = 1/R$ or $\mathbf{Z} = R = 8\ \text{k}\Omega$.

$$\mathbf{I}_o = \frac{\mathbf{V}}{\mathbf{Z}} = \frac{10\angle -90^\circ}{8000} = 1.25\angle -90^\circ\ \text{mA}$$

at $\omega = \omega_0$

$$P = I_o^2 R = (1.25x10^{-3})^2(8000) = 0.0125W = \boxed{12.5x10^{-3}W}$$

at $\omega = \omega_1,\ \omega_2$,

$$\boxed{P(\omega_o) = \frac{1}{2}\frac{V^2{}_m}{R}}$$

$$P = \frac{V^2}{2R} = \frac{10^2}{2x8000} = \boxed{6.25x10^{-3}W}$$